

**FERMIONIC SCREENING OPERATORS
IN THE SINE-GORDON MODEL**

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ABSTRACT. Extending our previous construction in the sine-Gordon model, we show how to introduce two kinds of fermionic screening operators, in close analogy with conformal field theory with $c < 1$.

1. INTRODUCTION

In our previous work [1], we have introduced and studied certain fermions which act on the space of local fields in the sine-Gordon model. The present note is intended as a supplement to that paper. Our aim here is to explain how to introduce fermionic analogs of the two screening operators well-known in CFT with $c < 1$.

In our notation, the Euclidean action of the sine-Gordon model is

$$(1.1) \quad \mathcal{A}^{\text{sG}} = \int \left\{ \left[\frac{1}{4\pi} \partial_z \varphi(z, \bar{z}) \partial_{\bar{z}} \varphi(z, \bar{z}) - \frac{\mu^2}{\sin \pi \beta^2} e^{-i\beta \varphi(z, \bar{z})} \right] - \frac{\mu^2}{\sin \pi \beta^2} e^{i\beta \varphi(z, \bar{z})} \right\} \frac{dz \wedge d\bar{z}}{2}.$$

In place of the coupling parameter β^2 , we shall henceforth use the parameter

$$(1.2) \quad \nu = 1 - \beta^2,$$

assuming $1/2 < \nu < 1$. We view the sine-Gordon model as a perturbation of the Liouville CFT. The latter is given by the action (1.1) without the last term, and the central charge is

$$(1.3) \quad c = 1 - \frac{6\nu^2}{1 - \nu}.$$

The exponential field parametrised as

$$(1.4) \quad \Phi_\alpha(z, \bar{z}) = e^{\frac{\nu}{2(1-\nu)} \alpha \{i\beta \varphi(z, \bar{z})\}}$$

turns in the CFT limit into a primary field with the scaling dimension

$$(1.5) \quad \Delta_\alpha = \frac{\nu^2}{4(1-\nu)} \alpha(\alpha - 2).$$

Abusing the language, we refer to (1.4) as a primary field in the sine-Gordon model as well.

In [1], we have considered two types of fermions. One of them, denoted $\beta_{2j-1}^*, \gamma_{2j-1}^*$ and $\bar{\beta}_{2j-1}^*, \bar{\gamma}_{2j-1}^*$, are local in nature. Acting on a primary field (1.4), together with the action of local integrals of motion, they create the space of all descendant fields, which in the CFT limit corresponds to the Verma module. The other type, denoted $\beta_{\text{screen},j}^*, \gamma_{\text{screen},j}^*$ and $\bar{\beta}_{\text{screen},j}^*, \bar{\gamma}_{\text{screen},j}^*$, are non-local. We call them the first fermionic screening operators. They relate primary fields with shifted exponents,

$$\Phi_{\alpha+2n\frac{1-\nu}{\nu}} \quad (n \in \mathbb{Z}_{\geq 0}),$$

as well as their corresponding descendants. For a more precise statement, see section 2.3 below. These are analogous to one of the two screening operators which exist in Liouville CFT. For these primary fields and fermionic descendants, the vacuum expectation values (VEVs) can be written explicitly. This is the main advantage of the fermionic description of the space of local fields.

As is well known, in CFT there exist another set of screening operators due to the symmetry

$$1 - \nu \longleftrightarrow \frac{1}{1 - \nu}.$$

The goal of this paper is to introduce their fermionic counterpart. We shall call them the second fermionic screening operators and denote them by $\beta_{\text{SCREEN},j}^*, \gamma_{\text{SCREEN},j}^*$ and $\bar{\beta}_{\text{SCREEN},j}^*, \bar{\gamma}_{\text{SCREEN},j}^*$.

The text is organised as follows. In Section 2, we consider the case of CFT on a cylinder. In [1], fermions are introduced by specifying their three point functions, which are encoded in a ‘structure function’ called ω^{sc} . In section 2.1 we re-examine the analytic structure of ω^{sc} and find its asymptotic behavior in sectorial domains. In section 2.2 we define the operators $\beta_{\text{SCREEN},j}^*$, etc., from the asymptotic expansion of ω^{sc} . They relate the primary fields

$$\Phi_{\alpha-2m\frac{1}{\nu}} \quad (m \in \mathbb{Z}_{\geq 0}).$$

We formulate the conjectural formulas pertaining to primary fields, and in section 2.3 for their descendants. In section 2.4 we verify various consistency relations required from the conjectures. In section 2.5 we glue together the two chiralities and check their three point functions against the Zamolodchikov-Zamolodchikov formula.

Section 3 is devoted to the case of the sine-Gordon model. Following the method in CFT we introduce the second fermionic screening operators by giving formulas for the VEV. In section 3.2 we check the formula with the Lukyanov-Zamolodchikov formula for one-point functions in the sine-Gordon model on the infinite plane. In section 3.3 we verify the validity of the consistency relations for them.

2. FERMIONIC SCREENING OPERATORS IN CFT

In this section we explain how to introduce the second fermionic screening operators in CFT.

2.1. Analytic structure of ω^{sc} . In [2] we considered chiral CFT on a cylinder $\mathbb{C}/2\pi i\mathbb{Z}$, with the central charge (1.3), and primary fields $\phi_\alpha(x)$ of scaling dimension (1.5). The objects of interest are the fermionic operators acting on the space of descendants of $\phi_\alpha(0)$. They are defined through the “structure function” $\omega^{\text{sc}}(\lambda, \mu|\kappa, \kappa', \alpha)$. It describes the three point functions of the fermionic descendants of $\phi_\alpha(0)$, in the presence of primary fields $\phi_{1-\kappa'}(-\infty)$, $\phi_{1+\kappa}(\infty)$ inserted at $x = \pm\infty$. In this subsection, we re-examine the analytic structure of ω^{sc} in order to extract the fermionic screening operators. So far the asymptotic analysis has been carried out only in the case $\kappa' = \kappa$. Throughout this text we shall make this assumption.

It is convenient to slightly modify ω^{sc} in [2], (11.5), and start from the function

$$(2.1) \quad \omega_0^{\text{sc}}(\lambda, \mu|\kappa, \kappa, \alpha) = \frac{1}{2\pi i} \iint dl dm \tilde{S}(l, \alpha) \tilde{S}(m, 2 - \alpha) \\ \times \Theta(l - i0, m|\kappa, \alpha) (e^{\pi i \nu + \delta} t)^{il} (e^{\pi i \nu + \delta} u)^{im}.$$

Here we have set

$$\lambda^2 = c(\nu, \kappa)t, \quad \mu^2 = c(\nu, \kappa)u, \\ c(\nu, \kappa) = \Gamma(\nu)^{-2} e^\delta \left(\frac{\nu\kappa}{2}\right)^{2\nu}, \quad \delta = -\nu \log \nu - (1 - \nu) \log(1 - \nu),$$

and

$$\tilde{S}(k, \alpha) = \frac{\Gamma(-ik + \frac{\alpha}{2}) \Gamma(\frac{1}{2} + i\nu k)}{\Gamma(-i(1 - \nu)k + \frac{\alpha}{2}) \sqrt{2\pi} (1 - \nu)^{(1-\alpha)/2}}.$$

The function $\Theta(l, m|\kappa, \alpha)$ is essentially the Mellin transform of the resolvent kernel of a linear integral operator associated with the Thermodynamic Bethe Ansatz equation. It has an asymptotic expansion in κ^{-2}

$$\Theta(l, m|\kappa, \alpha) \simeq \sum_{n=0}^{\infty} \Theta_n(l, m|\alpha) \kappa^{-2n}, \quad \Theta_0(l, m|\alpha) = \frac{-i}{l + m}.$$

The coefficients $\Theta_n(l, m|\alpha)$ ($n \geq 1$) are polynomials in l, m determined from the recursion relation in [2], Section 11. We remark that, when the fields $\phi_{1\mp\kappa}(\pm\infty)$ are changed to descendants, the same structure for ω_0^{sc} still persists, the only change being in $\Theta(l, m|\kappa, \alpha)$ (see e.g. [3]).

Since

$$(e^{\pi i \nu + \delta})^{ik} \tilde{S}(k, \alpha) \simeq \begin{cases} e^{-2\pi \nu k} & (k \rightarrow \infty), \\ 1 & (k \rightarrow -\infty), \end{cases}$$

the integral (2.1) converges term by term in κ^{-2} , provided t, u belong to the domain

$$(2.2) \quad \mathcal{S}_+ : -2\pi\nu < \arg t < 0.$$

By moving the contours into the lower half plane and taking residues in l, m , one obtains an asymptotic expansion of the form

$$(2.3) \quad \omega_0^{\text{sc}}(\lambda, \mu|\kappa, \kappa, \alpha) \simeq \sum_{j,k=1}^{\infty} t^{\alpha/2+j-1} u^{-\alpha/2+k} \omega_{0,j,k}^{\text{sc}}(\kappa, \alpha) \quad (t, u \rightarrow 0, t, u \in \mathcal{S}_+).$$

It is easy to see that the series in the right hand side is actually convergent on $D \times D$, where $D = \{t \in \mathbb{C} \mid |t| < 1\}$ (see Fig.1).

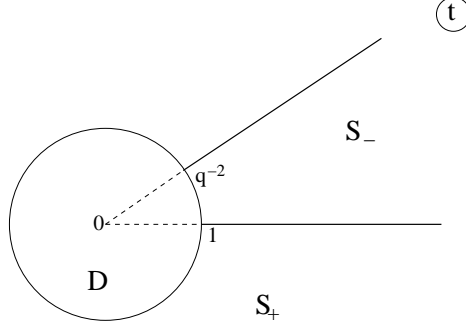


Fig.1: Domains of analyticity of ω_0^{sc} . This function is holomorphic in the unit disk D , and is continued analytically to the complex plane with the two cuts $[1, \infty)$ and $[q^{-2}, q^{-2}\infty)$ ($q = e^{\pi i \nu}$) being removed.

In fact, ω_0^{sc} can be continued analytically in each variable to the entire complex plane with the two cuts $[1, \infty)$, $[q^{-2}, q^{-2}\infty)$ ($q = e^{\pi i \nu}$) being removed. The analytic continuation in the variable t is achieved explicitly by a similar integral (2.1), wherein the function $\tilde{S}(l, \alpha)$ is changed to $\tilde{S}(l, \alpha)A_-(l, \alpha)$ with

$$A_-(l, \alpha) = -\frac{\sinh(\pi \nu l)}{\sinh \pi((1 - \nu)l + i\frac{\alpha}{2})} e^{\pi(l + i\frac{\alpha}{2})}.$$

To see this, notice that the series expansion (2.3) remains intact under this modification because

$$A_-(i(j - \frac{\alpha}{2}), \alpha) = 1 \quad (j \in \mathbb{Z}).$$

On the other hand, due to the behavior

$$A_-(l, \alpha) \simeq \begin{cases} e^{2\pi \nu l} & (l \rightarrow \infty) \\ e^{2\pi(1-\nu)l} & (l \rightarrow -\infty) \end{cases}$$

the domain of convergence of the integral is changed from \mathcal{S}_+ to

$$\mathcal{S}_- : 0 < \arg t < 2\pi(1 - \nu).$$

Of course the same procedure applies to the u variable as well. Altogether one obtains 4 different expressions of the same function ω_0^{sc} corresponding to the sectors $\mathcal{S}_\epsilon \times \mathcal{S}_{\epsilon'}$, $\epsilon, \epsilon' = \pm$. Let us write

$$\omega_0^{\text{sc}}(\lambda, \mu | \kappa, \kappa, \alpha) = \omega_{\epsilon, \epsilon'}^{\text{sc}}(\lambda, \mu | \kappa, \kappa, \alpha) + \omega_{0, \epsilon, \epsilon'}(\lambda / \mu | \alpha),$$

where

$$\begin{aligned} \omega_{\epsilon, \epsilon'}^{\text{sc}}(\lambda, \mu | \kappa, \kappa, \alpha) &= \frac{1}{2\pi i} \iint dl dm \tilde{S}_\epsilon(l, \alpha) \tilde{S}_{\epsilon'}(m, 2 - \alpha) \Theta(l + i0, m | \kappa, \alpha) (e^{\pi i \nu + \delta} t)^{il} (e^{\pi i \nu + \delta} u)^{im}, \\ \tilde{S}_\pm(k, \alpha) &= \tilde{S}(k, \alpha) A_\pm(k, \alpha), \quad A_+(k, \alpha) = 1, \end{aligned}$$

and

$$\omega_{0, \epsilon, \epsilon'}(\zeta | \alpha) = -i \int dl \zeta^{2il} \tilde{S}_\epsilon(l, \alpha) \tilde{S}_{\epsilon'}(-l, 2 - \alpha).$$

The function $\omega_{+,+}^{\text{sc}}$ coincides with ω^{sc} considered originally in [2].

When $\lambda^2, \mu^2 \rightarrow \infty$, the modified function $\omega_{\epsilon, \epsilon'}^{\text{sc}}$ has an asymptotic expansion in each sector. For instance, one has the expansions of the form

$$(2.4) \quad \omega_{+,+}^{\text{sc}}(\lambda, \mu | \kappa, \kappa, \alpha) \simeq \sum_{j,k=1}^{\infty} \lambda^{-\frac{2j-1}{\nu}} \mu^{-\frac{2k-1}{\nu}} \omega_{+,+,j,k}^{\text{sc}}(\kappa, \alpha),$$

$$(2.5) \quad \begin{aligned} \omega_{-,+}^{\text{sc}}(\lambda, \mu | \kappa, \kappa, \alpha) &\simeq \sum_{j,k=1}^{\infty} \lambda^{-\frac{2j-1}{\nu}} \mu^{-\frac{2k-1}{\nu}} \frac{e^{\frac{\pi i}{2\nu}(2j-1+\nu\alpha)}}{\cos \frac{\pi i}{2\nu}(2j-1+\nu\alpha)} \omega_{+,+,j,k}^{\text{sc}}(\kappa, \alpha) \\ &+ \sum_{j,k=1}^{\infty} \lambda^{-\frac{2j-\alpha}{1-\nu}} \mu^{-\frac{2k-1}{\nu}} \omega_{-,+,j,k}^{\text{sc}}(\kappa, \alpha), \end{aligned}$$

in $\mathcal{S}_+ \times \mathcal{S}_+$ and $\mathcal{S}_- \times \mathcal{S}_+$, respectively. The same quantity $\omega_{+,+,j,k}^{\text{sc}}(\kappa, \alpha)$ appears in two places.

Note that $\omega_{0, \epsilon, \epsilon'}(\zeta | \alpha)$ is independent of κ . Later on, in Section 3.1, we shall use their asymptotic expansions as $\zeta^2 \rightarrow \infty$:

$$\begin{aligned} \omega_{0,++}(\zeta | \alpha) &\simeq \sum_{j=1}^{\infty} \zeta^{-\frac{2j-1}{\nu}} t_{2j-1}(\alpha) + \sum_{j=1}^{\infty} \zeta^{-2j+\alpha} t'_j(\alpha), \\ \omega_{0,\mp\pm}(\zeta | \alpha) &\simeq \sum_{j=1}^{\infty} \zeta^{-\frac{2j-1}{\nu}} \frac{e^{\frac{\pi i}{2\nu}(2j-1\pm\nu\alpha)} t_{2j-1}(\alpha)}{\cos \frac{\pi}{2\nu}(2j-1\pm\nu\alpha)} + \sum_{j=1}^{\infty} \zeta^{-2j+\alpha} t'_j(\alpha), \\ \omega_{0,--}(\zeta | \alpha) &\simeq \sum_{j=1}^{\infty} \zeta^{-\frac{2j-1}{\nu}} \frac{e^{\frac{\pi i}{\nu}(2j-1)} t_{2j-1}(\alpha)}{\prod_{\pm} \cos \frac{\pi}{2\nu}(2j-1\pm\nu\alpha)} + \sum_{j=1}^{\infty} \zeta^{-2j+\alpha} t'_j(\alpha) + \sum_{j=1}^{\infty} \zeta^{-\frac{2j-\alpha}{1-\nu}} t''_j(\alpha), \end{aligned}$$

where

$$(2.6) \quad t_a(\alpha) = \frac{i}{\nu} \cot \frac{\pi}{2\nu}(a + \nu\alpha),$$

$$(2.7) \quad t'_j(\alpha) = -i \tan \frac{\pi\nu}{2}(2j - \alpha),$$

$$(2.8) \quad t''_j(\alpha) = \frac{i}{1-\nu} \tan \frac{\pi\nu(2j - \alpha)}{2(1-\nu)}.$$

The asymptotic expansions as $\zeta^2 \rightarrow 0$ are obtained from the equality

$$\omega_{0,\epsilon,\epsilon'}(\zeta|\alpha) = \omega_{0,\epsilon',\epsilon}(\zeta^{-1}|2-\alpha).$$

2.2. Second screening operators. The function $\tilde{S}(k, \alpha)$ has three series of poles and zeroes in k ,

$$(2.9) \quad \text{poles} \quad \frac{i(2j-1)}{2\nu}, \frac{i(2-2j-\alpha)}{2}; \quad \text{zeros} \quad \frac{i(2-2j-\alpha)}{2(1-\nu)},$$

where $j \in \mathbb{Z}_{\geq 1}$. In [2], the first is used to introduce the fermionic creation operators

$$\beta^*(\lambda) = \sum_{j=1}^{\infty} \lambda^{-\frac{2j-1}{\nu}} \beta_{2j-1}^*, \quad \gamma^*(\lambda) = \sum_{j=1}^{\infty} \lambda^{-\frac{2j-1}{\nu}} \gamma_{2j-1}^*,$$

while the second is used for the screening operators

$$\beta_{\text{screen}}^*(\lambda) = \sum_{j=1}^{\infty} \lambda^{2j-2+\alpha} \beta_{\text{screen},j}^*, \quad \gamma_{\text{screen}}^*(\lambda) = \sum_{j=1}^{\infty} \lambda^{2j-\alpha} \gamma_{\text{screen},j}^*.$$

They are related to the three point functions as follows.

$$\begin{aligned} \langle \beta^*(\lambda) \gamma^*(\mu) \rangle_{\kappa, \alpha} &= \omega^{\text{sc}}(\lambda, \mu | \kappa, \kappa, \alpha) \quad (\lambda, \mu \rightarrow \infty), \\ \langle \beta^*(\lambda) \gamma_{\text{screen}}^*(\mu) \rangle_{\kappa, \alpha} &= \omega^{\text{sc}}(\lambda, \mu | \kappa, \kappa, \alpha) \quad (\lambda \rightarrow \infty, \mu \rightarrow 0). \end{aligned}$$

Here and after we use the abbreviation

$$(2.10) \quad \langle \beta^*(\lambda) \gamma^*(\mu) \rangle_{\kappa, \alpha} = \frac{\langle 1 - \kappa | \beta^*(\lambda) \gamma^*(\mu) \phi_{\alpha}(0) | 1 + \kappa \rangle}{\langle 1 - \kappa | \phi_{\alpha}(0) | 1 + \kappa \rangle},$$

and so on. We shall refer to them as ‘pairings’ of fermions.

The analysis in the previous subsection motivates us to introduce yet another fermions

$$\begin{aligned} \beta_{-}^*(\lambda) &= \tilde{\beta}^*(\lambda) + \beta_{\text{SCREEN}}^*(\lambda), \quad \gamma_{-}^*(\lambda) = \tilde{\gamma}^*(\lambda) + \gamma_{\text{SCREEN}}^*(\lambda), \\ \tilde{\beta}^*(\lambda) &= \sum_{j=1}^{\infty} \lambda^{-\frac{2j-1}{\nu}} \frac{e^{\frac{\pi i}{2\nu}(2j-1+\nu\alpha)}}{\cos \frac{\pi}{2\nu}(2j-1+\nu\alpha)} \cdot \beta_{2j-1}^*, \\ \tilde{\gamma}^*(\lambda) &= \sum_{j=1}^{\infty} \lambda^{-\frac{2j-1}{\nu}} \frac{e^{\frac{\pi i}{2\nu}(2j-1-\nu\alpha)}}{\cos \frac{\pi}{2\nu}(2j-1-\nu\alpha)} \cdot \gamma_{2j-1}^*, \end{aligned}$$

such that together with $\beta_{+}^*(\lambda) = \beta^*(\lambda)$ and $\gamma_{+}^*(\lambda) = \gamma^*(\lambda)$ they satisfy

$$\langle \beta_{\epsilon}^*(\lambda) \gamma_{\epsilon'}^*(\mu) \rangle_{\kappa, \alpha} = \omega_{\epsilon, \epsilon'}^{\text{sc}}(\lambda, \mu | \kappa, \kappa, \alpha) \quad (\epsilon, \epsilon' = \pm),$$

for $\lambda, \mu \rightarrow \infty$. The operators

$$\beta_{\text{SCREEN}}^*(\lambda) = \sum_{j=1}^{\infty} \lambda^{-\frac{2j-\alpha}{1-\nu}} \beta_{\text{SCREEN},j}^*, \quad \gamma_{\text{SCREEN}}^*(\lambda) = \sum_{j=1}^{\infty} \lambda^{-\frac{2j-2+\alpha}{1-\nu}} \gamma_{\text{SCREEN},j}^*$$

correspond to the poles of $A_{-}(k, \alpha)$ which are not canceled by the zeros in (2.9).

We assign the following degrees to the Fourier components:

$$\begin{aligned} \deg \beta_{2j-1}^* &= 2j-1, & \deg \gamma_{2j-1}^* &= 2j-1, \\ \deg \beta_{\text{screen},j}^* &= -\nu(2j-2+\alpha), & \deg \gamma_{\text{screen},j}^* &= -\nu(2j-\alpha), \\ \deg \beta_{\text{SCREEN},j}^* &= \frac{\nu}{1-\nu}(2j-\alpha), & \deg \gamma_{\text{SCREEN},j}^* &= \frac{\nu}{1-\nu}(2j-2+\alpha). \end{aligned}$$

Their pairings are given explicitly as follows. Set

$$\begin{aligned} \tilde{D}_{2j-1}(\alpha) &= -\frac{\sqrt{i}}{\nu} \frac{1}{(j-1)!} \frac{\Gamma(\frac{1}{\nu}(j-\frac{1}{2})+\frac{\alpha}{2})}{\Gamma(\frac{1-\nu}{\nu}(j-\frac{1}{2})+\frac{\alpha}{2})}, \\ \tilde{E}_j(\alpha) &= -\frac{1}{\sqrt{i}} \frac{(-1)^j}{(j-1)!} \frac{\Gamma(\frac{1}{2}+\nu(j-\frac{\alpha}{2}))}{\Gamma(1-(1-\nu)j-\frac{\nu\alpha}{2})} \cdot e^{\pi i \nu(j-\frac{\alpha}{2})}, \\ \tilde{F}_j(\alpha) &= \frac{1}{\sqrt{i}} \frac{1}{1-\nu} \frac{(-1)^j}{(j-1)!} \frac{\Gamma(\frac{1}{2}-\frac{\nu}{1-\nu}(j-\frac{\alpha}{2}))}{\Gamma(1-\frac{1}{1-\nu}j+\frac{\nu\alpha}{2(1-\nu)})}. \end{aligned}$$

Then we have

$$\begin{aligned} (2.11) \quad \langle \beta_a^* \gamma_b^* \rangle_{\kappa, \alpha} &= \tilde{D}_a(\alpha) \tilde{D}_b(2-\alpha) \tilde{\Theta}(\frac{ia}{2\nu}, \frac{ib}{2\nu} | \kappa, \alpha), \\ \langle \beta_a^* \gamma_{\text{screen},k}^* \rangle_{\kappa, \alpha} &= \tilde{D}_a(\alpha) \tilde{E}_k(\alpha) \tilde{\Theta}(\frac{ia}{2\nu}, -i(k-\frac{\alpha}{2}) | \kappa, \alpha), \\ \langle \beta_{\text{SCREEN},j}^* \gamma_b^* \rangle_{\kappa, \alpha} &= \tilde{F}_j(\alpha) \tilde{D}_b(2-\alpha) \tilde{\Theta}(\frac{i}{1-\nu}(j-\frac{\alpha}{2}), \frac{ib}{2\nu} | \kappa, \alpha), \\ \langle \beta_{\text{SCREEN},j}^* \gamma_{\text{screen},k}^* \rangle_{\kappa, \alpha} &= \tilde{F}_j(\alpha) \tilde{E}_k(\alpha) \tilde{\Theta}(\frac{i}{1-\nu}(j-\frac{\alpha}{2}), -i(k-\frac{\alpha}{2}) | \kappa, \alpha). \end{aligned}$$

where

$$\tilde{\Theta}(l, m | \kappa, \alpha) = \Theta(l, m | \kappa, \alpha) x(\kappa)^{i(l+m)}, \quad x(\kappa) = \left(\frac{\nu\kappa}{2}\right)^{-2\nu} \Gamma(\nu)^2.$$

We have given only those pairings which will be relevant to our calculations.

We remark that the fermionic operators for the anti-chiral CFT are introduced by working with the function

$$\begin{aligned} (2.12) \quad \bar{\omega}_0^{\text{sc}}(\lambda, \mu | \kappa, \kappa, \alpha) &= \frac{1}{2\pi i} \int \int dl dm \tilde{S}(l, 2-\alpha) \tilde{S}(m, \alpha) \\ &\quad \times \Theta(l-i0, m | -\kappa, 2-\alpha) (e^{-\pi i \nu + \delta} t)^{il} (e^{-\pi i \nu + \delta} u)^{im}, \end{aligned}$$

where $c(\nu, \kappa)t = \lambda^{-2}$, $c(\nu, \kappa)u = \mu^{-2}$, as a counterpart for ω_0^{sc} . We have the following (conjectural) symmetry for the coefficients in the asymptotic expansions:

$$\Theta_n(l, m | 2-\alpha) = \Theta_n(m, l | \alpha).$$

This implies the symmetry between the first and the second chiralities: the roles of β_{2j-1}^* , $\beta_{\text{screen},j}^*$ and $\beta_{\text{SCREEN},j}^*$ are interchanged with $\bar{\gamma}_{2j-1}^*$, $\bar{\gamma}_{\text{screen},j}^*$ and $\bar{\gamma}_{\text{SCREEN},j}^*$,

respectively, and vice versa for γ^* and $\bar{\beta}^*$. The expansions at ∞ and 0 are interchanged:

$$\begin{aligned}\bar{\beta}^*(\lambda) &= \sum_{j=1}^{\infty} \lambda^{\frac{2j-1}{\nu}} \bar{\beta}_{2j-1}^*, \quad \bar{\gamma}^*(\lambda) = \sum_{j=1}^{\infty} \lambda^{\frac{2j-1}{\nu}} \bar{\gamma}_{2j-1}^*, \\ \bar{\beta}_{\text{screen}}^*(\lambda) &= \sum_{j=1}^{\infty} \lambda^{-2j+\alpha} \bar{\beta}_{\text{screen},j}^*, \quad \bar{\gamma}_{\text{screen}}^*(\lambda) = \sum_{j=1}^{\infty} \lambda^{-2j+2-\alpha} \bar{\gamma}_{\text{screen},j}^*, \\ \bar{\beta}_{\text{SCREEN}}^*(\lambda) &= \sum_{j=1}^{\infty} \lambda^{\frac{2j-2+\alpha}{1-\nu}} \bar{\beta}_{\text{SCREEN},j}^*, \quad \bar{\gamma}_{\text{SCREEN}}^*(\lambda) = \sum_{j=1}^{\infty} \lambda^{\frac{2j-\alpha}{1-\nu}} \bar{\gamma}_{\text{SCREEN},j}^*.\end{aligned}$$

However, since the relations between λ, μ and t, u are reversed in power, in terms of the integration variables l, m in (2.12), there is no interchange between the upper and lower half planes wherein we take residues. The domain of convergence for the integral representation of the function ω_0^{sc} is given by (2.2). It will change for $\bar{\omega}_0^{\text{sc}}$ because of the change

$$(2.13) \quad (e^{\pi i \nu + \delta} t)^{il} \rightarrow (e^{-\pi i \nu + \delta} t)^{il}$$

in the integrand. However, by the same reason as above, the domain of convergence in the variable λ^2 is unchanged. There is a sign change in the residues caused by the change (2.13). This will bring the pairings

$$\begin{aligned}\langle \bar{\beta}_a^* \bar{\gamma}_b^* \rangle_{\kappa, \alpha} &= \tilde{D}_a(2-\alpha) \tilde{D}_b(\alpha) \tilde{\Theta}(\frac{ib}{2\nu}, \frac{ia}{2\nu} | \kappa, \alpha), \\ \langle \bar{\beta}_{\text{screen},j}^* \bar{\gamma}_b^* \rangle_{\kappa, \alpha} &= -\tilde{E}'_j(\alpha) \tilde{D}_b(\alpha) \tilde{\Theta}(\frac{ib}{2\nu}, -i(j - \frac{\alpha}{2}) | \kappa, \alpha), \\ \tilde{E}'_k(\alpha) &= -\frac{1}{\sqrt{i}} \frac{(-1)^k}{(k-1)!} \frac{\Gamma(\frac{1}{2} + \nu(k - \frac{\alpha}{2}))}{\Gamma(1 - (1-\nu)k - \frac{\nu\alpha}{2})} \cdot e^{-\pi i \nu(k - \frac{\alpha}{2})}.\end{aligned}$$

The modification of the kernel function in the integral for the second screening operator in the anti-chiral case is $\tilde{S}(l, \alpha) \rightarrow \tilde{S}(l, \alpha) \bar{A}_-(l, \alpha)$ where

$$\bar{A}_-(l, \alpha) = -\frac{\sinh(\pi \nu l)}{\sinh \pi((1-\nu)l + i\frac{\alpha}{2})} e^{-\pi(l + i\frac{\alpha}{2})}.$$

This will bring

$$\begin{aligned}\bar{\beta}_-^*(\lambda) &= \tilde{\beta}^*(\lambda) + \bar{\beta}_{\text{SCREEN}}^*(\alpha), \quad \bar{\gamma}_-^*(\lambda) = \tilde{\gamma}^*(\lambda) + \bar{\gamma}_{\text{SCREEN}}^*(\alpha), \\ \tilde{\beta}^*(\lambda) &= \sum_{j=1}^{\infty} \lambda^{\frac{2j-1}{\nu}} \frac{e^{-\frac{\pi i}{2\nu}(2j-1+\nu\alpha)}}{\cos \frac{\pi}{2\nu}(2j-1+\nu\alpha)} \cdot \bar{\beta}_{2j-1}^*, \\ \tilde{\gamma}^*(\lambda) &= \sum_{j=1}^{\infty} \lambda^{\frac{2j-1}{\nu}} \frac{e^{-\frac{\pi i}{2\nu}(2j-1-\nu\alpha)}}{\cos \frac{\pi}{2\nu}(2j-1-\nu\alpha)} \cdot \bar{\gamma}_{2j-1}^*, \\ \langle \bar{\beta}_a^* \bar{\gamma}_{\text{SCREEN},k}^* \rangle_{\kappa, \alpha} &= -\tilde{D}_a(2-\alpha) \tilde{F}_k(\alpha) \tilde{\Theta}(\frac{i}{1-\nu}(k - \frac{\alpha}{2}), \frac{ia}{2\nu} | \kappa, \alpha), \\ \langle \bar{\beta}_{\text{screen},j}^* \bar{\gamma}_{\text{SCREEN},k}^* \rangle_{\kappa, \alpha} &= \tilde{E}'_j(\alpha) \tilde{F}_k(\alpha) \tilde{\Theta}(\frac{i}{1-\nu}(k - \frac{\alpha}{2}), -i(j - \frac{\alpha}{2}) | \kappa, \alpha).\end{aligned}$$

2.3. Fermionic descendants. Let us formulate our conjectures relating various primary fields with shifted exponents and their fermionic descendants. We follow the multi-index notation in [1] and write, for $I = \{i_1, \dots, i_p\}$ ($i_1 < \dots < i_p$),

$$\beta_I^* = \beta_{i_1}^* \cdots \beta_{i_p}^*, \quad \gamma_I^* = \gamma_{i_p}^* \cdots \gamma_{i_1}^*,$$

arranging β^* 's from left to right and γ^* 's in the opposite order. We also set

$$I(n) := \{1, 2, \dots, n\}, \quad I_{\text{odd}}(n) := \{1, 3, \dots, 2n-1\},$$

and write $aI + b$ to mean the set $\{ai + b \mid i \in I\}$.

For the first screening operators the conjectures in [1] state as follows. Let n be an non-negative integer, and let $I^\pm \subset 2\mathbb{Z}_{\geq 0} + 1$ be multi-indices such that $\#(I^+) = \#(I^-)$. Then

$$(2.14) \quad \phi_{\alpha+2n\frac{1-\nu}{\nu}}(0) \simeq c_{n,0}(\alpha) \beta_{I_{\text{odd}}(n)}^* \gamma_{\text{screen}, I(n)}^* \phi_\alpha(0),$$

$$(2.15) \quad \beta_{I^+}^* \gamma_{I^-}^* \phi_{\alpha+2n\frac{1-\nu}{\nu}}(0) \simeq c_{n,0}(\alpha) \beta_{I^++2n}^* \gamma_{I^--2n}^* \beta_{I_{\text{odd}}(n)}^* \gamma_{\text{screen}, I(n)}^* \phi_\alpha(0).$$

Here the sign \simeq means both sides have the same three point functions, e.g.,

$$\begin{aligned} & \langle 1 - \kappa | \beta_{I^+}^* \gamma_{I^-}^* \phi_{\alpha+2n\frac{1-\nu}{\nu}}(0) | 1 + \kappa \rangle \\ &= c_{n,0}(\alpha) \langle 1 - \kappa | \beta_{I^++2n}^* \gamma_{I^--2n}^* \beta_{I_{\text{odd}}(n)}^* \gamma_{\text{screen}, I(n)}^* \phi_\alpha(0) | 1 + \kappa \rangle. \end{aligned}$$

The main requirement is that in the two sides the κ dependent factors must coincide. The proportionality constants $c_{n,0}(\alpha)$ depend on the choice of normalisation of the primary fields. We shall fix them in the next subsection.

In the right hand side of (2.15), fermions with negative indices may appear. They are defined to be annihilation operators by enforcing the rule

$$(2.16) \quad \begin{aligned} \beta_{-a}^* &= t_a(2 - \alpha) \gamma_a, & \gamma_{-a}^* &= -t_a(\alpha) \beta_a, \\ [\beta_a, \beta_b^*]_+ &= \delta_{a,b}, & [\gamma_a, \gamma_b^*]_+ &= \delta_{a,b}, \end{aligned}$$

where $t_a(\alpha)$ is defined in (2.6).

Under the exchange $1 - \nu \leftrightarrow 1/(1 - \nu)$, β_{screen}^* and γ_{screen}^* interchange their roles with γ_{SCREEN}^* and β_{SCREEN}^* , respectively. Hence one expects the following relations:

$$(2.17) \quad \phi_{\alpha-2m\frac{1}{\nu}}(0) \simeq c_{0,m}(\alpha) \gamma_{I_{\text{odd}}(m)}^* \beta_{\text{SCREEN}, I(m)}^* \phi_\alpha(0),$$

$$(2.18) \quad \beta_{I^+}^* \gamma_{I^-}^* \phi_{\alpha-2m\frac{1}{\nu}}(0) \simeq c_{0,m}(\alpha) \beta_{I^+-2m}^* \gamma_{I^-+2m}^* \gamma_{I_{\text{odd}}(m)}^* \beta_{\text{SCREEN}, I(m)}^* \phi_\alpha(0).$$

Obviously the scaling dimensions match in both sides.

More generally, set

$$(2.19) \quad \alpha(n, m) = \alpha + 2n\frac{1-\nu}{\nu} - 2m\frac{1}{\nu}.$$

Combining (2.14)–(2.18) together and using (2.16), one finds that

$$(2.20) \quad \beta_{I^+}^* \gamma_{I^-}^* \phi_{\alpha(n,m)}(0) \simeq c_{n,m}(\alpha) \begin{cases} \beta_{I^++2(n-m)}^* \gamma_{I^--2(n-m)}^* \beta_{I_{\text{odd}}(n-m)}^* \phi_\alpha^{(n,m)}(0) & (n > m), \\ \beta_{I^+}^* \gamma_{I^-}^* \phi_\alpha^{(n,m)}(0) & (n = m), \\ \beta_{I^+-2(m-n)}^* \gamma_{I^-+2(m-n)}^* \gamma_{I_{\text{odd}}(m-n)}^* \phi_\alpha^{(n,m)}(0) & (n < m), \end{cases}$$

where we have set

$$\phi_\alpha^{(n,m)}(0) = \beta_{\text{SCREEN},I(m)}^* \gamma_{\text{screen},I(n)}^* \phi_\alpha(0).$$

Note that, in the case $m = n$, the β_{2j-1}^* , γ_{2j-1}^* drop out altogether from the primary field, leaving

$$\phi_{\alpha(m,m)}(0) \simeq c_{m,m}(\alpha) \phi_\alpha^{(m,m)}(0).$$

2.4. Consistency check for descendants. Formula (2.20) requires some consistency conditions, which determine $c_{n,m}(\alpha)$ once we fix $c_{1,0}(\alpha)$ and $c_{0,1}(\alpha)$. We show (modulo computer checks) that if we fix $c_{1,0}(\alpha)$, $c_{0,1}(\alpha)$ and $c_{1,1}(\alpha)$, then, $c_{n,m}(\alpha)$ is given by

$$(2.21) \quad c_{n,m}(\alpha) = \begin{cases} (-1)^{m(n-m)} \prod_{j=0}^{n-m-1} c_{1,0}(\alpha(m+j, m)) \prod_{j=0}^{m-1} c_{1,1}(\alpha(j, j)) & \text{if } n \geq m; \\ (-1)^{n(m-n)} \prod_{j=0}^{m-n-1} c_{0,1}(\alpha(n, n+j)) \prod_{j=0}^{n-1} c_{1,1}(\alpha(j, j)) & \text{if } n \leq m. \end{cases}$$

Recall the reduction rule

$$\begin{aligned} & \mathcal{F}(\beta_a^*, \gamma_b^*, \beta_{\text{SCREEN},j}^*, \gamma_{\text{screen},k}^*) \phi_{\alpha(n,m)} \\ & \simeq \mathcal{F}(\beta_{a+2(n-m)}^*, \gamma_{b+2(m-n)}^*, \beta_{\text{SCREEN},j+m}^*, \gamma_{\text{screen},k+n}^*) \\ & \quad \times \begin{cases} \beta_{I_{\text{odd}}}^*(n-m) & (n \geq m) \\ \gamma_{I_{\text{odd}}}^*(m-n) & (n \leq m) \end{cases} \cdot \beta_{\text{SCREEN},I(m)}^*, \gamma_{\text{screen},I(n)}^* \phi_\alpha, \end{aligned}$$

where \mathcal{F} is a monomial of the fermions. There are consistency conditions among the three quantities $c_{1,0}(\alpha)$, $c_{0,1}(\alpha)$ and $c_{1,1}(\alpha)$. One can reduce $\phi_{\alpha(1,1)}$ to ϕ_α in three different ways. First we have

$$\phi_{\alpha(1,1)} \simeq c_{1,1}(\alpha) \beta_{\text{SCREEN},1}^* \gamma_{\text{screen},1}^* \phi_\alpha$$

We have also

$$\begin{aligned} \phi_{\alpha(1,1)} &= c_{1,0}(\alpha(0,1)) \beta_1^* \gamma_{\text{screen},1}^* \phi_{\alpha(0,1)} \\ &= c_{1,0}(\alpha(0,1)) c_{0,1}(\alpha) \beta_{-1}^* \gamma_{\text{screen},1}^* \gamma_1^* \beta_{\text{SCREEN},1}^* \phi_\alpha \\ &= \frac{i}{\nu} \cot \frac{\pi}{2\nu} (1 - \nu\alpha) c_{1,0}(\alpha(0,1)) c_{0,1}(\alpha) \beta_{\text{SCREEN},1}^* \gamma_{\text{screen},1}^* \phi_\alpha \end{aligned}$$

This and a similar calculation in a reversed order give the compatibility condition

$$\begin{aligned} c_{1,1}(\alpha) &= \frac{i}{\nu} \cot \frac{\pi}{2\nu} (1 - \nu\alpha) c_{1,0}(\alpha(0,1)) c_{0,1}(\alpha) \\ &= \frac{i}{\nu} \cot \frac{\pi}{2\nu} (1 + \nu\alpha) c_{0,1}(\alpha(1,0)) c_{1,0}(\alpha). \end{aligned}$$

A solution to the compatibility condition is given by

$$\begin{aligned} c_{1,0}(\alpha) &= 1 / \cos \pi \left(\frac{\alpha}{2} + \frac{1}{2\nu} \right), \\ c_{0,1}(\alpha) &= \sin \pi \left(-\frac{\alpha}{2} + \frac{1}{2\nu} \right). \end{aligned}$$

Now let us discuss $c_{n,m}(\alpha)$ to be given by (2.21). Let us choose $I^+, I^- \subset 2\mathbb{Z}_{\geq 0} + 1$ such that

$$\#(I^+) = \#(I^-) = L,$$

and consider the descendant $\beta_{I^+}^* \gamma_{I^-}^* \phi_{\alpha(n,m)}$. Let us use the abbreviation

$$\langle * \rangle_\kappa = \langle 1 - \kappa | * | 1 + \kappa \rangle.$$

One can compute the ratio of 3 point functions $\langle \beta_{I^+}^* \gamma_{I^-}^* \phi_{\alpha(n,m)} \rangle_\kappa / \langle \phi_\alpha \rangle_\kappa$ in two different ways. For instance, if $n \geq m$, then (2.20) gives the consistency condition

$$\begin{aligned} & \frac{\langle \beta_{I^+}^* \gamma_{I^-}^* \phi_{\alpha(n,m)} \rangle_\kappa}{\langle \phi_{\alpha(n,m)} \rangle_\kappa} \prod_{j=0}^{n-m-1} \frac{\langle \phi_{\alpha(m+j+1,m)} \rangle_\kappa}{\langle \phi_{\alpha(m+j,m)} \rangle_\kappa} \prod_{j=0}^{m-1} \frac{\langle \phi_{\alpha(j+1,j+1)} \rangle_\kappa}{\langle \phi_{\alpha(j,j)} \rangle_\kappa} \\ &= c_{n,m}(\alpha) \frac{\langle \beta_{I^+ + 2(n-m)}^* \gamma_{I^- - 2(n-m)}^* \beta_{I_{\text{odd}}(n-m)}^* \gamma_{\text{SCREEN}, I(m)}^* \gamma_{\text{screen}, I(n)}^* \phi_\alpha \rangle_\kappa}{\langle \phi_\alpha \rangle_\kappa}, \end{aligned}$$

and similarly for $m \leq n$. One can apply (2.20) to each term in the product of the left hand side. Then, calculation of the two sides by Wick contraction leads to a family of identities involving the functions $\tilde{\Theta}(l, m|\kappa, \alpha)$ and $\tilde{D}_j(\alpha)$, $\tilde{E}_j(\alpha)$, $\tilde{F}_j(\alpha)$.

Two remarks are in order. First, it is easy to see that the consistency condition is unchanged if we replace $\tilde{\Theta}(l, m|\kappa, \alpha)$ with $\Theta(l, m|\kappa, \alpha)$. This is because $x(\kappa)$ appears with the level as exponent. So, we forget $x(\kappa)$ in the following calculation. Next, we have performed computer checks with various choices of n, m that the consistency condition reduces to that of $\kappa = \infty$. We did such checks up to order κ^{-6} . Note that

$$\Theta(l, m|\infty, \alpha) = -\frac{i}{l+m}.$$

In the following we use only this specialization, and write $\Theta(l, m|\infty, \alpha)$ simply as $\Theta(l, m)$.

In order to write out these relations systematically, we introduce the following index sets:

$$\begin{aligned} I_{>}^\pm(n, m) &= \{j \pm 2(n-m) | j \in I^\pm\} \cap \mathbb{Z}_{>0}, \\ I_c^\pm(n, m) &= \{-j \mp 2(n-m) | j \in I^\pm\} \cap \mathbb{Z}_{>0}, \\ \tilde{I}^\pm(n, m) &= (I_{\text{odd}}(\pm(n-m)) \setminus I_c^\mp(n, m)) \sqcup I_{>}^\pm(n, m), \end{aligned}$$

where $I_{\text{odd}}(n)$ is the empty set if $n \leq 0$. The subscript c in $I_c^\pm(n, m)$ signifies the fermion contraction. Note that these index sets are actually depend only on the difference $n - m$.

For example, $I_c^\pm(0, 0) = \emptyset$, $\tilde{I}^\pm(0, 0) = I^\pm$; if $I^+ = \{1, 3\}$ and $I^- = \{1, 5\}$ we have $\tilde{I}^+(2, 0) = \{1, 5, 7\}$, $\tilde{I}^-(2, 0) = \{1\}$, $I_c^+(2, 0) = \emptyset$, $I_c^-(2, 0) = \{3\}$. Note that if $n \geq m$, then $I_c^+(n, m) = \emptyset$; if $n \leq m$, then $I_c^-(n, m) = \emptyset$.

Set

$$\vartheta_{n,m} = \sum_{2k-1 \in I_c^+(n,m) \sqcup I_c^-(n,m)} (k-1),$$

$$t_{n,m}^\pm(\alpha) = \prod_{a \in I_c^\pm(n,m)} \frac{i}{\nu} \cot \frac{\pi}{2\nu} (a \mp \nu\alpha).$$

We define the Cauchy determinant with normalization factors

$$\begin{aligned} & \mathcal{D}_{I^+, I^-}(n, m | \alpha) \\ &= \det \begin{pmatrix} \left(\Theta\left(\frac{ia}{2\nu}, \frac{ib}{2\nu}\right) \right)_{a \in \tilde{I}^+(n,m), b \in \tilde{I}^-(n,m)} & \left(\Theta\left(\frac{ia}{2\nu}, -\frac{i(2k-\alpha)}{2}\right) \right)_{a \in \tilde{I}^+(n,m), k \in I(n)} \\ \left(\Theta\left(\frac{i(2j-\alpha)}{2(1-\nu)}, \frac{ib}{2\nu}\right) \right)_{j \in I(m), b \in \tilde{I}^-(n,m)} & \left(\Theta\left(\frac{i(2j-\alpha)}{2(1-\nu)}, -\frac{i(2k-\alpha)}{2}\right) \right)_{j \in I(m), k \in I(n)} \end{pmatrix} \\ & \times \prod_{a \in \tilde{I}^+(n,m)} \tilde{D}_a(\alpha) \prod_{a \in \tilde{I}^-(n,m)} \tilde{D}_a(2-\alpha) \prod_{j \in I(m)} \tilde{F}_j(\alpha) \prod_{k \in I(n)} \tilde{E}_k(\alpha). \end{aligned}$$

In this notation, we have

$$\frac{\langle \beta_{I^+}^* \gamma_{I^-}^* \phi_{\alpha(n,m)}(0) \rangle_{\kappa=\infty}}{\langle \phi_\alpha(0) \rangle_{\kappa=\infty}} = c_{n,m}(\alpha) \begin{cases} (-1)^{L(n-m)+\vartheta_{n,m}} t_{n,m}^-(\alpha) \mathcal{D}_{I^+, I^-}(n, m | \alpha) & (n \geq m); \\ (-1)^{(L+1)(m-n)+\vartheta_{n,m}} t_{n,m}^+(\alpha) \mathcal{D}_{I^+, I^-}(n, m | \alpha) & (n \leq m). \end{cases}$$

On the other hand, the left hand side of the consistency relation reads

$$\begin{aligned} & \frac{\langle \beta_{I^+}^* \gamma_{I^-}^* \phi_{\alpha(n,m)}(0) \rangle_\infty}{\langle \phi_{\alpha(n,m)}(0) \rangle_\infty} = \mathcal{D}_{I^+, I^-}(0, 0 | \alpha(n, m)), \\ & \frac{\langle \phi_{\alpha(n,m)}(0) \rangle_\infty}{\langle \phi_\alpha(0) \rangle_\infty} = \prod_{j=1}^N \Theta\left(i \frac{j-\frac{\alpha}{2}}{1-\nu}, -i \left(j - \frac{\alpha}{2}\right)\right) \prod_{j=1}^{m-N} \Theta\left(\frac{i}{1-\nu} \left(1 - \frac{\alpha(N+N+j)}{2}\right), i \frac{1}{2\nu}\right) \\ & \times \prod_{j=1}^{n-N} \Theta\left(i \frac{1}{2\nu}, -i \left(1 - \frac{\alpha(N+j, N)}{2}\right)\right) \prod_{j=0}^{N-1} \tilde{E}_1(\alpha(j, j)) \tilde{F}_1(\alpha(j, j)) \\ & \times \prod_{j=0}^{n-N-1} \tilde{D}_1(\alpha(N+j, N)) \tilde{E}_1(\alpha(N+j, N)) \\ & \times \prod_{j=0}^{m-N-1} (-1) \tilde{D}_1(2-\alpha(N, N+j)) \tilde{F}_1(\alpha(N, N+j)) \\ & \times \begin{cases} \prod_{j=0}^{n-m-1} c_{1,0}(\alpha(m+j, m)) \prod_{j=0}^{m-1} c_{1,1}(\alpha(j, j)) & \text{if } n \geq m; \\ \prod_{j=0}^{m-n-1} c_{0,1}(\alpha(n, n+j)) \prod_{j=0}^{n-1} c_{1,1}(\alpha(j, j)) & \text{if } n \leq m, \end{cases} \end{aligned}$$

where $N = \min(n, m)$. The formula (2.21) for the normalization factor $c_{n,m}(\alpha)$ is obtained from these relations.

The relations for the normalization factor (2.21) are the same for the anti-chiral case. This follows from the formulas for the fermionic pairings that are given in Section 2.2.

2.5. Gluing two chiralities. The primary field in the full CFT is

$$\Phi_\alpha(0) = S(\alpha)\phi_\alpha(0)\bar{\phi}_\alpha(0)$$

with some numerical coefficient $S(\alpha)$.

Introduce the screened primary field by

$$(2.22) \quad \Phi_\alpha^{(n,m)}(0) = s_{n,m}(\alpha)\beta_{\text{SCREEN},I(m)}^*\bar{\beta}_{\text{screen},I(n)}^*\gamma_{\text{screen},I(n)}^*\bar{\gamma}_{\text{SCREEN},I(m)}^*\Phi_\alpha(0),$$

$$s_{n,m}(\alpha) = \mu^{2n - \frac{2}{1-\nu}m} \left\{ \prod_{k=1}^n t'_k(\alpha) \prod_{j=1}^m t''_j(\alpha) \right\}^{-1}.$$

Gluing together the relation (2.20) with their anti-chiral partners, one finds that

$$(2.23) \quad \Phi_{\alpha(n,m)}(0) \cong C_{n,m}(\alpha) \times \begin{cases} \beta_{I_{\text{odd}}(n-m)}^* \bar{\gamma}_{I_{\text{odd}}(n-m)}^* \Phi_\alpha^{(n,m)}(0) & (n > m), \\ \Phi_\alpha^{(m,m)}(0) & (n = m), \\ \bar{\beta}_{I_{\text{odd}}(m-n)}^* \gamma_{I_{\text{odd}}(m-n)}^* \Phi_\alpha^{(n,m)}(0) & (n < m), \end{cases}$$

where the normalisation coefficient $C_{n,m}(\alpha)$ is related to $c_{n,m}(\alpha)$ in (2.20) and its anti-chiral counterpart $\bar{c}_{n,m}(\alpha)$ via

$$C_{n,m}(\alpha)s_{n,m}(\alpha) = \frac{S(\alpha(n,m))}{S(\alpha)}c_{n,m}(\alpha)\bar{c}_{n,m}(\alpha).$$

The recursive relations, which determine the general case out of $C_{0,1}, C_{1,0}, C_{1,1}$, are

$$C_{n,m}(\alpha) = \begin{cases} C_{n,m-1}(\alpha)C_{0,1}(\alpha(n,m-1)) & (n < m), \\ C_{m-1,m-1}(\alpha)C_{1,1}(\alpha(m-1,m-1)) & (n = m), \\ C_{n-1,m}(\alpha)C_{1,0}(\alpha(n-1,m)) & (n > m). \end{cases}$$

These relations are deduced from the corresponding ones for $c_{n,m}(\alpha), \bar{c}_{n,m}(\alpha)$ and the simple relations

$$t'_n(\alpha) = t'_1(\alpha(n-1,m)), \quad t''_m(\alpha) = t''_1(\alpha(n,m-1)).$$

Choosing the ‘exponential normalisation’ $\Phi_\alpha(z, \bar{z})\Phi_{-\alpha}(0) = |z|^{-4\Delta_\alpha}(1 + O(|z|^2))$, one can determine $C_{n,m}(\alpha)$ by comparing with the Liouville three point functions by Zamolodchikov-Zamolodchikov [4].

This calculation for the coefficient $C_{1,0}(\alpha)$ is explained in [1], eq.(6.8). For $C_{0,1}(\alpha)$, the relevant formula can be obtained by the substitution $b \rightarrow b^{-1}$ in [1], (A.2) (which amounts to $-1/\nu \rightarrow (1-\nu)/\nu$), and comparing the result with the pairing $\langle \beta_{\text{SCREEN},1}^* \gamma_1^* \rangle_{\kappa,\alpha}$ given by (2.11).

We have checked that the κ -dependent part agrees up to the degree κ^{-6} . Collecting the coefficients, we obtain the following result.

$$(2.24) \quad C_{1,0}(\alpha) = \Gamma(\nu)^{2\alpha+2\frac{1-\nu}{\nu}} i\nu \tan \frac{\pi}{2\nu} (\nu\alpha + 1) \\ \times \frac{\Gamma\left(\frac{1}{2} - \frac{\alpha}{2} - \frac{1-\nu}{2\nu}\right)}{\Gamma\left(\frac{1}{2} + \frac{\alpha}{2} + \frac{1-\nu}{2\nu}\right)} \frac{\Gamma\left(\frac{\alpha}{2} + \frac{1-\nu}{2\nu}\right)}{\Gamma\left(-\frac{\alpha}{2} - \frac{1-\nu}{2\nu}\right)} \frac{\Gamma(-1 + \nu - \nu\alpha)}{\Gamma(1 - \nu + \nu\alpha)},$$

$$(2.25) \quad C_{0,1}(\alpha) = -\{(1-\nu)^{2\nu}\Gamma(\nu)^{-2}\}^{\frac{\nu\alpha-1}{\nu(1-\nu)}} i\nu \tan \frac{\pi}{2\nu} (\nu\alpha - 1) \\ \times \frac{\Gamma\left(\frac{1}{2} - \frac{\alpha}{2} + \frac{1}{2\nu}\right)}{\Gamma\left(\frac{1}{2} + \frac{\alpha}{2} - \frac{1}{2\nu}\right)} \frac{\Gamma\left(\frac{\alpha}{2} - \frac{1}{2\nu}\right)}{\Gamma\left(-\frac{\alpha}{2} + \frac{1}{2\nu}\right)} \frac{\Gamma\left(-\frac{1}{1-\nu} + \frac{\nu\alpha}{1-\nu}\right)}{\Gamma\left(\frac{1}{1-\nu} - \frac{\nu\alpha}{1-\nu}\right)},$$

and

$$(2.26) \quad C_{1,1}(\alpha) = \{(1-\nu)^2\Gamma(\nu)^{-2}\}^{\frac{\nu(\alpha-1)}{1-\nu}} \times \frac{\Gamma\left(\frac{\nu}{1-\nu}(\alpha-1)\right)}{\Gamma\left(-\frac{\nu}{1-\nu}(\alpha-1)\right)} \frac{\Gamma(-\nu(\alpha-1))}{\Gamma(\nu(\alpha-1))}.$$

The third one can be obtained from the first two by the relation

$$C_{1,1}(\alpha) = -\frac{1}{\nu^2} \cot^2 \frac{\pi}{2} (\alpha - \frac{1}{\nu}) C_{0,1}(\alpha) C_{1,0}(\alpha - \frac{2}{\nu}).$$

3. SINE-GORDON MODEL

In this section, we discuss the sine-Gordon model. Since the working is entirely similar to the one of [1], we shall be rather brief, indicating only the necessary formulas.

3.1. Fermions in sine-Gordon model. As in the CFT case, the fermions in the sine-Gordon model are defined by giving the expectation values of basic descendants. The way of introducing the second screening operators is similar to that in CFT: in place of the function $\omega_R^{\text{sG}}(\zeta, \xi|\alpha)$ given in [1], eq. (7.8), we use the following formal integral

$$\omega_{R,\epsilon,\epsilon'}^{\text{sG}}(\zeta, \xi|\alpha) = -\frac{\pi i}{2} \iint \frac{dl}{2\pi} \frac{dm}{2\pi} \zeta^{2il} \xi^{2im} \Theta_R^{\text{sG}}(l, m|\alpha) \\ \times A_\epsilon(l, \alpha) \frac{e^{-\pi\nu l}}{\cosh \pi\nu l} A_{\epsilon'}(m, 2-\alpha) \frac{e^{-\pi\nu m}}{\cosh \pi\nu m},$$

which we understand as the expansions in ζ^2, ξ^2 at $\infty, 0$; the expansions are obtained by taking residues in l, m in the upper/lower half planes. Accordingly, we define $\beta_\epsilon^{+*}(\zeta), \gamma_\epsilon^{+*}(\zeta)$ ($\epsilon = \pm$) for $\zeta^2 \rightarrow \infty$, and $\beta_\epsilon^{-*}(\zeta), \gamma_\epsilon^{-*}(\zeta)$ ($\epsilon = \pm$) for $\zeta^2 \rightarrow 0$ as

follows:

$$\begin{aligned}
\beta_\epsilon^{+*}(\zeta) &= \begin{cases} \beta^*(\mu\zeta) + \bar{\beta}_{\text{screen}}^*(\mu^{-1}\zeta) & (\epsilon = +); \\ \tilde{\beta}^*(\mu\zeta) + \beta_{\text{SCREEN}}^*(\mu\zeta) + \bar{\beta}_{\text{screen}}^*(\mu^{-1}\zeta) & (\epsilon = -), \end{cases} \\
\gamma_\epsilon^{+*}(\zeta) &= \begin{cases} \gamma^*(\mu\zeta) + \bar{\gamma}_{\text{screen}}^*(\mu^{-1}\zeta) & (\epsilon = +); \\ \tilde{\gamma}^*(\mu\zeta) + \gamma_{\text{SCREEN}}^*(\mu\zeta) + \bar{\gamma}_{\text{screen}}^*(\mu^{-1}\zeta) & (\epsilon = -), \end{cases} \\
\beta_\epsilon^{-*}(\zeta) &= \begin{cases} \bar{\beta}^*(\mu^{-1}\zeta) + \beta_{\text{screen}}^*(\mu\zeta) & (\epsilon = +); \\ \tilde{\beta}^*(\mu^{-1}\zeta) + \bar{\beta}_{\text{SCREEN}}^*(\mu^{-1}\zeta) + \beta_{\text{screen}}^*(\mu\zeta) & (\epsilon = -), \end{cases} \\
\gamma_\epsilon^{-*}(\zeta) &= \begin{cases} \bar{\gamma}^*(\mu^{-1}\zeta) + \gamma_{\text{screen}}^*(\mu\zeta) & (\epsilon = +); \\ \tilde{\gamma}^*(\mu^{-1}\zeta) + \bar{\gamma}_{\text{SCREEN}}^*(\mu^{-1}\zeta) + \gamma_{\text{screen}}^*(\mu\zeta) & (\epsilon = -). \end{cases}
\end{aligned}$$

The prescription for the fermion pairings is as follows.

$$\begin{aligned}
\langle \beta_\epsilon^{\pm*}(\zeta) \gamma_{\epsilon'}^{\pm*}(\xi) \rangle_R^{\text{sG}} &= \omega_{R,\epsilon,\epsilon'}^{\text{sG}}(\zeta, \xi | \alpha), \\
\langle \beta_\epsilon^{\pm*}(\zeta) \gamma_{\epsilon'}^{\mp*}(\xi) \rangle_R^{\text{sG}} &= \omega_{R,\epsilon,\epsilon'}^{\text{sG}}(\zeta, \xi | \alpha) + \omega_{0,\epsilon,\epsilon'}(\zeta/\xi | \alpha).
\end{aligned}$$

It is immediate to write them in terms of the Fourier components. For that purpose let us introduce the shorthand notation

$$\hat{\beta}_a^* = \begin{cases} \mu^{-\frac{a}{\nu}} \beta_a^* & (a > 0), \\ -\mu^{\frac{a}{\nu}} \bar{\beta}_{-a}^* & (a < 0), \end{cases} \quad \hat{\gamma}_a^* = \begin{cases} \mu^{-\frac{a}{\nu}} \gamma_a^* & (a > 0), \\ -\mu^{\frac{a}{\nu}} \bar{\gamma}_{-a}^* & (a < 0), \end{cases}$$

$$\hat{\beta}_{\text{SCREEN},j}^* = \begin{cases} \mu^{-\frac{2j-\alpha}{1-\nu}} \beta_{\text{SCREEN},j}^* & (j > 0), \\ -\mu^{\frac{2j-\alpha}{1-\nu}} \bar{\beta}_{\text{SCREEN},1-j}^* & (j \leq 0). \end{cases}$$

$$\hat{\gamma}_{\text{SCREEN},j}^* = \begin{cases} \mu^{-\frac{2j-2+\alpha}{1-\nu}} \gamma_{\text{SCREEN},j}^* & (j > 0), \\ -\mu^{\frac{2j-2+\alpha}{1-\nu}} \bar{\gamma}_{\text{SCREEN},1-j}^* & (j \leq 0). \end{cases}$$

The minus sign in front of μ comes from the orientation of cycles for complex integration.

For all $a, b \in 2\mathbb{Z} + 1$ and $j, k \in \mathbb{Z}$ we have

$$\langle \hat{\beta}_a^* \hat{\gamma}_b^* \rangle_R^{\text{sG}} = \frac{i}{2\pi} \frac{1}{\nu^2} \left\{ \Theta_R^{\text{sG}} \left(\frac{ia}{2\nu}, \frac{ib}{2\nu} \middle| \alpha \right) + \delta_{a+b,0} \text{sgn}(a) 2\pi i \nu^2 t_a(\alpha) \right\},$$

$$\langle \hat{\beta}_a^* \hat{\gamma}_{\text{SCREEN},k}^* \rangle_R^{\text{sG}} = \frac{i}{2\pi} \frac{1}{\nu} \Theta_R^{\text{sG}} \left(\frac{ia}{2\nu}, \frac{i(2k+\alpha-2)}{2(1-\nu)} \middle| \alpha \right) t_k''(2-\alpha),$$

$$\langle \hat{\beta}_{\text{SCREEN},j}^* \hat{\gamma}_b^* \rangle_R^{\text{sG}} = \frac{i}{2\pi} \frac{1}{\nu} t_j''(\alpha) \Theta_R^{\text{sG}} \left(\frac{i(2j-\alpha)}{2(1-\nu)}, \frac{ib}{2\nu} \middle| \alpha \right),$$

$$\begin{aligned}
\langle \hat{\beta}_{\text{SCREEN},j}^* \hat{\gamma}_{\text{SCREEN},k}^* \rangle_R^{\text{sG}} &= \frac{i}{2\pi} t_j''(\alpha) t_k''(2-\alpha) \left\{ \Theta_R^{\text{sG}} \left(\frac{i(2j-\alpha)}{2(1-\nu)}, \frac{i(2k+\alpha-2)}{2(1-\nu)} \middle| \alpha \right) \right. \\
&\quad \left. - \delta_{j+k,1} \text{sgn}(2j-1) 2\pi i t_j''(\alpha)^{-1} \right\}.
\end{aligned}$$

The first screening operators decouple from the rest:

$$(3.1) \quad \langle \bar{\beta}_{\text{screen},j}^* \gamma_{\text{screen},k}^* \rangle_R^{\text{sG}} = \delta_{j,k} \mu^{-2(2j-\alpha)} t'_j(\alpha),$$

$$(3.2) \quad \langle \beta_{\text{screen},j}^* \bar{\gamma}_{\text{screen},k}^* \rangle_R^{\text{sG}} = \delta_{j,k} \mu^{-2(2j-2+\alpha)} t'_j(2-\alpha).$$

It is not surprising that the symmetry between two kinds of screening operators is broken: the sG model is a perturbation of CFT by the operators which behaves differently with respect to them.

3.2. One point functions on the plane. In the limit $R \rightarrow \infty$, ω_R^{sG} vanishes, and the only non-trivial pairings become

$$\begin{aligned} \langle \beta_a^* \bar{\gamma}_a^* \rangle_\infty^{\text{sG}} &= \mu^{\frac{2}{\nu}} \frac{i}{\nu} \cot \frac{\pi}{2\nu} (a + \nu\alpha), & \langle \bar{\beta}_a^* \gamma_a^* \rangle_\infty^{\text{sG}} &= \mu^{\frac{2}{\nu}} \frac{i}{\nu} \cot \frac{\pi}{2\nu} (a - \nu\alpha), \\ \langle \beta_{\text{SCREEN},j}^* \bar{\gamma}_{\text{SCREEN},j}^* \rangle_\infty^{\text{sG}} &= \mu^{\frac{2}{1-\nu} (2j-\alpha)} t''_j(\alpha), \\ \langle \bar{\beta}_{\text{screen},j}^* \gamma_{\text{screen},j}^* \rangle_\infty^{\text{sG}} &= \mu^{-2(2j-\alpha)} t'_j(\alpha). \end{aligned}$$

Here again we have listed only the pairings which will be used later.

The screened primary field is defined by the same formula (2.22) as before. Notice that it is normalized as

$$\frac{\langle \Phi_\alpha^{(n,m)}(0) \rangle_\infty^{\text{sG}}}{\langle \Phi_\alpha(0) \rangle_\infty^{\text{sG}}} = \mu^{-2n(n-\alpha) + \frac{2m(m-\alpha)}{1-\nu}}.$$

In [1], we have verified for $(n, m) = (1, 0)$ that this formula agrees with the Lukyanov-Zamolodchikov formula [5] for the one-point function on the plane. Let us check that it works also for the second screening operators, i.e., for $(n, m) = (0, 1)$.

From (2.22) and the pairings above we calculate

$$\frac{\langle \Phi_{\alpha-\frac{2}{\nu}}(0) \rangle_\infty^{\text{sG}}}{\langle \Phi_\alpha(0) \rangle_\infty^{\text{sG}}} = \mu^{\frac{2}{\nu} + \frac{2(1-\alpha)}{1-\nu}} C_{0,1}(\alpha) \frac{i}{\nu} \cot \frac{\pi}{\nu} (1 - \nu\alpha).$$

On the other hand, a direct computation with the formula in [5] gives

$$\begin{aligned} \frac{\langle \Phi_{\alpha-\frac{2}{\nu}}(0) \rangle_\infty^{\text{sG}}}{\langle \Phi_\alpha(0) \rangle_\infty^{\text{sG}}} &= (\mu \Gamma(\nu))^{\frac{2(1-\nu\alpha)}{\nu(1-\nu)}} H_{0,1} \left(\frac{\alpha}{2} - \frac{1}{2\nu} \right), \\ H_{0,1}(x) &= -(1-\nu)^{\frac{4\nu}{1-\nu}x} \frac{\Gamma(x) \Gamma(\frac{1}{2} - x)}{\Gamma(-x) \Gamma(\frac{1}{2} + x)} \cdot \frac{\Gamma(\frac{2\nu}{1-\nu}x)}{\Gamma(-\frac{2\nu}{1-\nu}x)}. \end{aligned}$$

We find perfect agreement with the formula above.

3.3. Consistency. Finally let us touch upon the consistency conditions.

As explained in [1], section 9, the following identities can be derived from the linear integral equation defining $\Theta_R^{\text{sG}}(l, m|\alpha)$.

$$\begin{aligned} & \Theta_R^{\text{sG}}\left(l, m \middle| \alpha \pm \frac{2(1-\nu)}{\nu}\right) \cdot \left(\Theta_R^{\text{sG}}\left(\pm \frac{i}{2\nu}, \mp \frac{i}{2\nu} \middle| \alpha\right) - 2\pi\nu \cot \frac{\pi}{2\nu}(1 \pm \nu\alpha) \right) \\ &= \left| \begin{array}{cc} \Theta_R^{\text{sG}}\left(\pm \frac{i}{2\nu}, \mp \frac{i}{2\nu} \middle| \alpha\right) - 2\pi\nu \cot \frac{\pi}{2\nu}(1 \pm \nu\alpha) & \Theta_R^{\text{sG}}\left(\pm \frac{i}{2\nu}, m \mp \frac{i}{\nu} \middle| \alpha\right) \\ \Theta_R^{\text{sG}}\left(l \pm \frac{i}{\nu}, \mp \frac{i}{2\nu} \middle| \alpha\right) & \Theta_R^{\text{sG}}\left(l \pm \frac{i}{\nu}, m \mp \frac{i}{\nu} \middle| \alpha\right) \end{array} \right|, \\ & \Theta_R^{\text{sG}}\left(l, m \middle| \alpha \mp 2\right) \cdot \left(\Theta_R^{\text{sG}}\left(\frac{i(1-\alpha \pm 1)}{2(1-\nu)}, -\frac{i(1-\alpha \pm 1)}{2(1-\nu)} \middle| \alpha\right) \mp 2\pi(1-\nu) \cot \frac{\pi\nu(1-\alpha \pm 1)}{2(1-\nu)} \right) \\ &= \left| \begin{array}{cc} \Theta_R^{\text{sG}}\left(\frac{i(1-\alpha \pm 1)}{2(1-\nu)}, -\frac{i(1-\alpha \pm 1)}{2(1-\nu)} \middle| \alpha\right) \mp 2\pi(1-\nu) \cot \frac{\pi\nu(1-\alpha \pm 1)}{2(1-\nu)} & \Theta_R^{\text{sG}}\left(\frac{i(1-\alpha \pm 1)}{2(1-\nu)}, m \middle| \alpha\right) \\ \Theta_R^{\text{sG}}\left(l, -\frac{i(1-\alpha \pm 1)}{2(1-\nu)} \middle| \alpha\right) & \Theta_R^{\text{sG}}\left(l, m \middle| \alpha\right) \end{array} \right|. \end{aligned}$$

Combining them together and manipulating with determinants, one deduces further that for all $r, s \geq 0$

$$(3.3) \quad \Theta_R^{\text{sG}}\left(l, m \middle| \alpha - \frac{2r}{\nu} + 2s\frac{1-\nu}{\nu}\right) = \det \begin{pmatrix} \mathcal{A} & \mathcal{B} & \mathbf{x} \\ \mathcal{C} & \mathcal{D} & \mathbf{y} \\ {}^t\mathbf{u} & {}^t\mathbf{v} & z \end{pmatrix} / \det \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{A}_{j,k} &= \Theta_R^{\text{sG}}\left(\frac{i(2j-\alpha)}{2(1-\nu)}, -\frac{i(2k-\alpha)}{2(1-\nu)} \middle| \alpha\right) - \delta_{j,k} 2\pi(1-\nu) \cot \frac{\pi\nu(2j-\alpha)}{2(1-\nu)}, \\ \mathcal{B}_{j,b} &= \Theta_R^{\text{sG}}\left(\frac{i(2j-\alpha)}{2(1-\nu)}, -\frac{ib}{2\nu} \middle| \alpha\right), \quad \mathcal{C}_{a,k} = \Theta_R^{\text{sG}}\left(\frac{ia}{2\nu}, -\frac{i(2k-\alpha)}{2(1-\nu)} \middle| \alpha\right), \\ \mathcal{D}_{a,b} &= \Theta_R^{\text{sG}}\left(\frac{ia}{2\nu}, -\frac{ib}{2\nu} \middle| \alpha\right) - \delta_{a,b} \text{sgn}(a) 2\pi\nu \cot \frac{\pi(a+\nu\alpha)}{2\nu}, \\ \mathbf{x}_j &= \Theta_R^{\text{sG}}\left(\frac{i(2j-\alpha)}{2(1-\nu)}, m + i\frac{r-s}{\nu} \middle| \alpha\right), \quad \mathbf{y}_a = \Theta_R^{\text{sG}}\left(\frac{ia}{2\nu}, m + i\frac{r-s}{\nu} \middle| \alpha\right), \\ \mathbf{u}_k &= \Theta_R^{\text{sG}}\left(l - i\frac{r-s}{\nu}, -\frac{i(2k-\alpha)}{2(1-\nu)} \middle| \alpha\right), \quad \mathbf{v}_b = \Theta_R^{\text{sG}}\left(l - i\frac{r-s}{\nu}, -\frac{ib}{2\nu} \middle| \alpha\right), \\ z &= \Theta_R^{\text{sG}}\left(l - i\frac{r-s}{\nu}, m + i\frac{r-s}{\nu} \middle| \alpha\right), \end{aligned}$$

and the indices range over

$$j, k \in I(r), \quad a, b \in \begin{cases} I_{\text{odd}}(s-r) & (s \geq r), \\ -I_{\text{odd}}(r-s) & (r \geq s). \end{cases}$$

Appropriate specialisations of (3.3) guarantee the consistency of the formula (2.23). Since the argument is the same as in [1], we omit the details.

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